STOCHASTIC SCHRODINGER/ HEISENBERG EQUATION vs. LANGEVIN EQUATION

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Key References

1. Ford, Kac, Mazur; Jour. Math. Phys. 6 (1966), 504

2. Ford, Lewis, O'Connell; Phys. Rev. A, 37(11) (1988), 4419.

3. Li, Ford, O'Connell, Phys. Rev. A, 41(10), (1990), 5287.

4. SINHA, K.B., Jour. IND. INSTT. OF SCIENCE 77 (1997), 275

5. SINHA, Probability towards 2000, L.N. Statistics # 128 (1998), Springer.

6. Goswami & Sinha, Quantum Stochastic Processes and Non-Comm. Geometry, C.U.P. 2007.

In Classical Mechanics, the Hamilton's eqn (in 1-dim for simplicity):

$\frac{\partial H}{\partial p} = \dot{q}$, $-\frac{\partial H}{\partial q} = \dot{p}$, this pair

describes only those motions where energy is conserved.

How to understand evolution in a "dissipative systems" or "systems with frictional forces" of which there are many in nature?

Langevin suggested adding an adhoc random force along with the usual impressed force. Then the above eqns will look like:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} + E(t),$$ where

$E(t)$ is the 'random' force with correlation:

$\langle E(t) E(t') \rangle = \sigma^2 \delta(t-t')$, σ^2 a +ve constant, and $\langle \cdot \rangle$ is the expectation.

Q: Can one derive or understand this eqn (Langevin Equation) on the basis of Hamiltonian mechanics (Classical and Quantum) ?

The first serious attempt was by FKM (ref 1). The model is one of a large no. of harmonic oscillators in which a distinguished one is the "observed system" (OS) while the rest model the "environment" (E) or heat bath, interacting amongst each other by a quadratic term. Kac himself later said: "... either the Langevin equation here is a fluke of the special FKM model or there is no general valid quantum Langevin equation".

In ref(2), a very similar attempt is made with independent oscillator model, the coupling this time is linear, and a few nuclear approximations are made to obtain a Langevin equation:

$$\ddot{q} + 2b\dot{q} + V'(q) = E(t) \text{ with}$$

$$\frac{1}{2}\langle E(t)E(t') + E(t')E(t)\rangle \xrightarrow[\hbar \to 0]{} 4kTb\,\delta(t-t')$$

in the classical limit.

Criticisms: We find that an Hamiltonian model with artificial approximations can lead to just about anything. More specifically (1) the derivations are approximate & mathematically crude (2) More importantly, what is the real purpose of such an eqn?

If we had considered the total system of OS + E, then clearly the total evolution would be reversible and non-dissipative. But we are not interested in this, we want to know only the "projected" or "reduced" motion of the OS by "washing out" the effects of the E, and that is clearly expected to be dissipative.

Classical Damped Harmonic Oscillator

wellknown eq^n of motion:

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \nu^2 x = 0 \quad (k>0,\ \nu>k). \quad (1)$$

In analogy with Hamilton's eq^{ns} (though there is no real Hamiltonian describing this motion), we write

$$\frac{d\tilde{x}}{dt} = \delta\tilde{p} - k\tilde{x}\ ; \quad \frac{d\tilde{p}}{dt} = -\delta\tilde{x} - k\tilde{p} \quad \cdots (2)$$

where $\delta = \sqrt{\nu^2 - k^2}$, $\tilde{x} = x$, $\tilde{p} = \delta^{-1} p$.

Solve & observe that the Poisson Bracket

$\{x(t), p(t)\} = -e^{-2kt}$ though $\{x(0), p(0)\} = -1$.

Model: the particle in the harmonic potential $\frac{1}{2}\nu^2 x^2$ is coupled with a classical Brownian motion ω representing the environment and its motion is given by:

$$\left.\begin{aligned} d\tilde{x} &= \sqrt{2k}\,\tilde{p}\,d\omega + (-k\tilde{x} + \delta\tilde{p})\,dt \\ d\tilde{p} &= -\sqrt{2k}\,\tilde{x}\,d\omega + (-k\tilde{p} - \delta\tilde{x})\,dt, \end{aligned}\right\} \cdots (3)$$

so that if we average out w.r.t the Br. motion, the averaged observables clearly satisfy (2) and hence (1). What about the solution of (3) which will give the evolution of the observables $\tilde{x}, \tilde{p}$ themselves. They will clearly be random. Luckily in this simple case, we can exactly solve it:

$$(4) \quad \begin{cases} x(t) = x(0)\cos\left(\delta t + \sqrt{2\beta}\,\omega(t)\right) + \delta^{-1} p(0) \sin\left(\delta t + \sqrt{2\beta}\,\omega(t)\right) \\ p(t) = p(0) \cos\left(\delta t + \sqrt{2\beta}\,\omega(t)\right) - \delta x(0) \sin\left(\delta t + \sqrt{2\beta}\,\omega(t)\right) \end{cases}$$

conclusions:

(i) $\{x(t), p(t)\} = \{x(0), p(0)\} = -1$ for all t and almost all ω. Moreover, $\delta^2 x^2 + p^2 = \text{Const}$ $\forall$ t and ω, i.e. $(x(0), p(0)) \longmapsto (x(t), p(t))$ is a random symplectic flow

(ii) As said before, the expections $\langle x \rangle$ & $\langle p \rangle$ satisfy the eqns of motion (2) we had started with.

(iii) The motion is essentially one of a random one on the ellipse $\delta^2 x^2 + p^2 =$ constant.

Is such a situation generic?

NO!

For a general Hamiltonian $H=\frac{1}{2}p^2+V(x)$, we can mimic the above procedure by introducing a modified Hamiltonian $H' = H-\frac{1}{2}b^2x^2$ and replacing (3) by

$$\left.\begin{aligned} dx &= \sqrt{2b}\,p\,d\omega + (-bx + \{H', x\})\,dt \\ dp &= -\sqrt{2b}\,x\,d\omega + (-bp + \{H', p\})\,dt. \end{aligned}\right\} \quad \text{--(5)}$$

Though a simple calculation using Ito formula gives $\{x(t), p(t)\} = \{x(0), p(0)\} = -1$ $\forall t$ and ω, the unsatisfactory feature of the model lies in the fact that H' is not bounded below (and hence not physically nice) for most potentials V.

Dissipative Quantum system

Since the attempt of modifying H doesn't work, the extra term $-\frac{1}{2}b^2x^2$ in H' has to be compensated some other way. One possible way is to allow introduction of quantum noise or quantum Brownian motion.

Solution of Equation (5)

Let $H' = p^2/2 + V(x) - \frac{b^2}{2} x^2$ and let $\{X \equiv X(t, x(0), p(0))$ and $P \equiv P(t, x(0), p(0))\}$ be the solution of the Hamilton's eqn:

$$\frac{dX}{dt} = \{H', X\} \text{ \& } \frac{dP}{dt} = -\{H', P\}. \quad \cdots (5')$$

Then the solution of (5) is given by:

$$\left.\begin{aligned} x &= X\cos\theta + P\sin\theta \\ p &= P\cos\theta - X\sin\theta \end{aligned}\right\}, \qquad (5'')$$

where $\theta(t) = \sqrt{2b}\,\omega(t)$. Direct computation easily verifies this. Then it is also easy to verify:

(i) $\{x, p\} = \{X, P\} = \{x(0), p(0)\}$
$= -1.$

(ii) Again the motion is a random symplectic flow. But it is a singular one because H' is not bounded below in many situations and solution may be unstable.

~~No! Not in general~~.

This is entirely due to the presence of the term $+b^2x$ in:

~~$\dot{x} = p - bx, \quad \dot{p} =$~~

$$dx = (p - bx)\,dt + \sqrt{2b}\, p\, dw(t)$$

$$dp = \left(-bp + b^2x - V'(x)\right) dt + \sqrt{2}\, b x\, dw(t).$$

Unlike in the oscillator case, where $V(x) = \frac{1}{2}\omega^2 x^2$, $V'(x)$ will not compensate for $b^2 x$.

Quantum Theory of Noise

In $h_p \otimes L^2(\mathbb{R}_2)$

$$\simeq h_p \otimes \Gamma(L^2(\mathbb{R}_+, \mathbb{C}^2))$$

we look at the equation

$$dU(t) = U(t)\left(\sum_{j=1}^{2}\{L_j^* \, dA_j(t) - L_j \, dA_j^*(t) - \tfrac{1}{2} L_j^* L_j \, dt\} + iH\,dt\right)$$

if $L_j = 0 \Rightarrow$ Schrodinger eqn.

where $L_1 = \sqrt{2}b\left(\frac{p - ix}{\sqrt{2}}\right)$, $L_2 = \exp\left(\frac{ib^2x^2}{2}\right)$,

$H = \frac{1}{2}p^2 + V(x)$, $U(0) = I$.

Remark: Under suitable conditions on V, does there exist unitary solutions of the above "stochastic Schrodinger equation"?

If $L_j = 0 \; \forall j$, this question is equivalent to that about the self-adjointness of H and has been investigated thoroughly and to a great extent, satisfactorily in the last 50 years.

If $L_j \neq 0$, the situation is more tricky. The first results, in this direction are by Mohari (Ph.D. thesis, I.S.I. Delhi 1991), Fagnola (Preprint 1991), Fagnola-Chebotarev (J. Func. Anal. 153(2), 1998), Mohari-Sinha (Proc. Ind. Acad. Sc.-Maths., 102(3), 1992). For a general introduction to Quantum Stochastic Calculus, the reference are the book by K.R. Parthasarathy. (Birkhauser 1992) and reference 6 (the book by Goswami-Sinha).

More recently, some further developements in this problem has happened: (1) Lindsay-Sinha, On holomorphic Co-cycles, unpublished (2) Das, Goswami, Sinha, Homomorphic Flows and Trotter Product for Stochastic flows, under revision for Comm. Math. Phys.

Continuous Time Stochastic Processes

As in the above, we want to lift the familiar classical continuous time processes like standard Brownian motion (SBM) & Poisson process onto a bigger 'pot'; but once we have done that the 'pot' naturally will have more stuff.

The mathematical structure necessary for this is that of a Fock space. Let h be a separable Hilbert space.

$$\Gamma \equiv \Gamma(h) \equiv \Gamma_{symm}(h) \equiv \bigoplus_{n=0}^{\infty} h^{(\otimes n)}, \quad \text{where}$$

$h^{(\otimes 0)} = \mathbb{C}$, $h^{(\otimes n)}$ is the symmetrised n-fold tensor product of h, and the infinite direct sum has its natural inner product and norm. $\Gamma \equiv \Gamma(h)$ is called the Fock space over h.

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A convenient *total* set in Γ consists of the *exponential vectors* $\{e(f) \mid f \in h_p\}$ given as:

$$e(f) = 1 \oplus f \oplus \frac{f^{(2)}}{\sqrt{2!}} \oplus \cdots \oplus \frac{f^{(n)}}{\sqrt{n!}} \oplus \cdots,$$ leading to

$$\langle e(f), e(g) \rangle_{\Gamma} = \exp\left(\langle f, g \rangle_{h_p}\right).$$ This explains the name 'exponential vectors'.

For our specific purposes, we'll take $h_p = L^2(\mathbb{R}_+)$ and then note the following properties.

1. Γ has continuous time tensor decomposition property. Set $\Gamma_t = \Gamma(L^2([0,t]))$, $\Gamma^t = \Gamma(L^2([t,\infty)))$; then Γ is unitarily isomorphic to the Hilbert tensor product $\Gamma_t \otimes \Gamma^t$. For this, note that for the exponential vector $e(f)$

$$e(f) \simeq e(f\chi_{[0,t]}) \otimes e(f\chi_{[t,\infty)})$$ and that each piece generate a total set in each space

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2. Consider the (locally convex) topological space $C[0,\infty)$, the space of continuous functions on $\mathbb{R}_+$, vanishing at 0, with the topology of uniform convergence on compact sets in $\mathbb{R}_+$. Wiener constructed a (Gaussian) probability measure $\mathbb{P}$ (Wiener measure) on $C[0,\infty)$. It is a fact:

$$\int \overline{\mathcal{E}(f)}\,\mathcal{E}(g)\,d\mathbb{P} = \langle e(f), e(g)\rangle_\Gamma \ , \text{ where}$$

$$\mathcal{E}(f) = \exp\left\{ \int_0^\infty f(s)\,d\omega(s) - \tfrac{1}{2}\int_0^\infty f^2(s)\,ds \right\} ; \quad f, g \in L^2(\mathbb{R}_+).$$

In the above, $\omega(\cdot) \in C[0,\infty)$ (the BM) and the integral $\int_0^\infty f(s)\,d\omega(s)$ (Wiener integral) has to be understood as a limit of the random variable. $\sum_{\{j\}\in\Pi} f(s_j)\{\omega(s_{j+1}) - \omega(s_j)\}$ where $\{0 \le s_1 \le \cdots \le s_j < s_{j+1} \cdots\} \equiv \Pi$ (a partition), in the Hilbert space $L^2(C([0,\infty)), \mathbb{P})$ as the width of the partition $|\Pi| \to 0$.

Then the above equality establishes an unitary isomorphism between the Wiener L^2-space, $L^2(C([0,\infty)), \mathbb{P})$ and $\Gamma(L^2(\mathbb{R}_+))$.

<u>Remarks</u>: (i) In many applications, we may need another Hilbert space h_0, called the initial space or system space and we write

$$\mathcal{H} \equiv h_0 \otimes \Gamma, \quad \mathcal{H}_t \equiv h_0 \otimes \Gamma_t, \quad \mathcal{H}^t \equiv \Gamma^t, \text{ leading}$$

to an unitary isomorphism $\mathcal{H} \longmapsto \mathcal{H}_t \otimes \mathcal{H}^t$, $t \geq 0$

(ii) In Γ, the vector $e(0)$, called <u>vacuum</u>, plays a special role and is isomorphically mapped to the function 1 in $L^2(C([0,\infty)), \mathbb{P})$.

3. For any Hilbert space h_0, if we set $\mathcal{B}(h_0)$ to be the (C^*-algebra) of linear bounded operators (defined everywhere), then one can set up an increasing filtration (for $t \uparrow \infty$) of sub-algebras of $\mathcal{B}(\mathcal{H})$ as follows:

Write $\mathcal{B}_t \equiv \mathcal{B}(\mathcal{H}_t) \otimes I^t$ ← identity operator in $\mathcal{H}^t$,

and note that $\mathcal{B}_t \uparrow \mathcal{B}_\infty \equiv \mathcal{B}(\mathcal{H})$.

Remarks. In classical language, the filtration of σ-algebras for SBM's is generated by the "increments of the SBM's ≤ t". If we think of the SBM as an (unbounded) multiplication operator by the 'independent variable' the path ω(·), then the classical filtration corresponds to the filtration of sub algebras of $\{\mathcal{B}_t\}_{t \geq 0}$ generated by these multiplication ops. upto time t.

4. There are some special operators (martingales) in this theory which correspond to the classical stochastic processes.

Annihilation Op: $\forall f, g \in L^2(\mathbb{R}_+)$,

$$a(g)\, e(f) = \langle g, f \rangle_{L^2(\mathbb{R}_+)}\, e(f)$$ → " e(f)'s are eigenvectors of these ops."

Creation Op.:

$$a^+(g)\, e(f) = -i \frac{d}{d\varepsilon}\, e(f+\varepsilon g)\Big|_{\varepsilon=0} \rightarrow \text{these}$$

look like "the infitesmal generators of the translation group of $L^2(\mathbb{R}_+)$."

If we set $\mathcal{E} = \text{lin. span}\{e(f) \mid f \in L^2(\mathbb{R}_+)\}$,

then one can verify that:

$$\langle a^+(g)e(f), e(h)\rangle = \langle e(f), a(g)e(h)\rangle,$$

$$[a(f), a^+(g)] = \langle f, g\rangle I_\Gamma \quad \text{on } \mathcal{E}.$$

Conservation Op.: Let $A \in \mathcal{B}(L^2(\mathbb{R}_+))$ and set

$$\lambda(A)\, e(f) = i \frac{d}{d\varepsilon}\, e(\exp(-iA\varepsilon)f)\Big|_{\varepsilon=0} - \text{generator}$$

of a semigroup action in $L^2(\mathbb{R}_+)$.

One has:

$$[\lambda(A_1), \lambda(A_2)] = \lambda([A_1, A_2])$$

$$[a(f), \lambda(A)] = a(A^* f)$$

$$[a^+(f), \lambda(A)] = -a^+(Af),$$

5. <u>The Basic Martingales:</u> Let D_0 and M be dense linear subsets of h and $L^2(\mathbb{R}_+)$ resp. such that $\chi_{[s,t]} f \in M$ for all $f \in M$ and $0 \le s \le t < \infty$.

A family $\{X_t\}_{t \geq 0}$ of operators in $\mathcal{H}$ is said to be <u>adapted</u> w.r.t. (D_0, M) if and

(a) $\text{Dom}(X_t) \supseteq D_0 \otimes \mathcal{E}(M)$

(b) $X_t\left(u e(f \chi_{[0,t]})\right) \in \mathcal{H}_t$ for every $u \in D_0$, $f \in M$.

<u>Definitions</u> Annihilation Process: $A(t) = a(\chi_{[0,t]})$,

Creation Process: $A^{\dagger}(t) = a^{\dagger}(\chi_{[0,t]})$,

Conservation Process: $\Lambda(t) = \lambda(\text{Mult. by } \chi_{[0,t]})$.

An adapted process $\{X_t\}_{t \geq 0}$ is said to be a martingale if for $s < t$; $u, v \in D_0$; $f \in M$;

$$\langle u e(f \chi_{[0,s]}), X_t v e(g \chi_{[0,s]}) \rangle = \langle u e(f_{[0,s]}), X_s v e(g \chi_{[0,s]}) \rangle .$$

From the definitions above, it is clear that $\{A(t), A^{+}(t), \Lambda(t)\}_{t\geq}$ are adapted and are martingales.

<u>Remarks</u>: <u>Identification with Classical Processes</u>

Set $q(t) \equiv \dfrac{A(t)+A^{+}(t)}{\sqrt{2}}$, $p(t) \equiv \dfrac{A(t)-A^{+}(t)}{i\sqrt{2}}$.

Then it can be seen that $p(t)$ and $q(t)$ are unitary conjugate of each other in Γ and under the unitary isomorphism between Γ and the Wiener space $L^{2}(\mathbb{P})$,

$q(t) \longmapsto$ multiplication by the SBM $\omega(\cdot)$ in $L^{2}(\mathbb{P})$

and $p(t)$ is ~~also~~ unitarily equivalent to the same operator. However,

$$[p(s), q(t)] = -i\min(t,s),$$

i.e. a pair of non-commuting SBM's are living in the Fock space.

Next, set $X_\lambda(t) \equiv \Lambda(t) + \sqrt{\lambda}\,\big(A(t) + A^+(t)\big) + \lambda t$ for some $\lambda > 0$ and observe that

a) $X_\lambda(t)$ is essentially self-adjoint on $\mathcal{E}$

b) If we define the Weyl operators (unitary) in Γ by

$$W(f)\, e(g) = \exp\left(-\tfrac{1}{2}\|f\|^2 - \langle f, g\rangle\right) e(f+g),$$

then

$$X_\lambda(t) = W\left(-\sqrt{\lambda}\, \chi_{[0,t]}\right) \Lambda(t)\, W\left(\sqrt{\lambda}\, \chi_{[0,t]}\right)$$

and one can easily compute the expectation value of $\exp(i\alpha X_\lambda(t))$ in the vacuum state:

$\langle e(0), \exp(i\alpha X_\lambda(t))\, e(0)\rangle$ and find this to be

$$= \exp\left\{ \lambda t \left(e^{i\alpha} - 1\right)\right\} \quad \forall\, \alpha \in \mathbb{R},\ t \geq 0,$$

This we recognize as the same as the characteristic function (sort of Fourier transform) of the classical Poisson process leading to the identification of $\{X_\lambda(t)\}$ as the Poisson with intensity λ.

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Thus, the filtration $\{\mathcal{B}_t\}$ in Γ contains, in particular, 3 commutative sub-algebras: $\{\varphi(q(t)) \otimes I^t\}$, $\{\varphi(p(t)) \otimes I^t\}$, $\{\varphi(X_\lambda(t)) \otimes I^t\}$ (for $\varphi \in L^\infty(\mathbb{R})$) which are mutually non-commuting. But they are all adapted to the same filtration $\{\mathcal{B}_t\}_{t \geq 0}$!!

6. Stochastic (non-commutative) integration:

$\{M_t\}$ be any of the 3 basic martingales $\{A(t), A^\dagger(t), \Lambda(t)\}$; then for $0 \leq s \leq t < \infty$,

$$(M_t - M_s) e(f) = e(f \chi_{[0,s]}) \otimes \{(M_t - M_s) e(f \chi_{[s,t]})\} \otimes e(f \chi$$

i.e. the increments of M "lives in Γ_t^s, between s and t". Thus as in the classical theory, one defines the integral for an adapted simple process $\{L_t\}_{t \geq 0}$ w.r.t. the M_t- increments on $D_0 \otimes \mathcal{E}(\mathcal{M})$ and then extend the definition

The first theorem:

$$\left\langle u e(f), \int_0^t L(s) M(ds)\, v e(g) \right\rangle =$$

$$\int_0^t \langle u e(f), L(s)\, v e(g)\rangle \begin{Bmatrix} g(s) \\ \bar f(s) \\ \bar f(s) g(s)\end{Bmatrix} ds,$$ according as

$M = A, A^{+}$ or Λ respectively.

The Second Theorem (Ito formula):

$$\left\langle \int_0^t L_1(s)\begin{Bmatrix}A(ds)\\A^{+}(ds)\\\Lambda(ds)\end{Bmatrix} u e(f), \int_0^t L_2(s) \begin{Bmatrix}A(ds)\\A^{+}(ds)\\\Lambda(ds)\end{Bmatrix} v e(g)\right\rangle$$

$$= \int_0^t ds \left\langle L_1(s) u e(f), \left(\int_0^s L_2(\tau) M(d\tau)\right) v e(g)\right\rangle \begin{Bmatrix} \bar f(s)\\ g(s) \\ \bar f(s) g(s)\end{Bmatrix} +$$

$$\int_0^t ds \left\langle \int_0^s L_1(\tau) M(d\tau)\, u e(f), L_2(s)\, v e(g)\right\rangle \begin{Bmatrix} g(s)\\ \bar f(s)\\ \bar f(s) g(s)\end{Bmatrix} +$$

$$\int_0^t ds \langle L_1(s) u e(f), L_2(s) v e(g)\rangle \begin{bmatrix} 0&0&0\\0&1&g(s)\\0&\bar f(s)&\bar f(s)g(s)\end{bmatrix}.$$

Ito Correction

Ito Correction table

dM \ dM	dA	$dA^\dagger$	$d\Lambda$
dA	0	dt	dA
$dA^\dagger$	0	0	0
$d\Lambda$	0	$dA^\dagger$	$d\Lambda$

7. Stochastic Differential Equation

$$dX = X\left(L\,dA^\dagger + K\,dA + F\,d\Lambda + G\,dt\right)$$

with initial value $X_{t=0} = X_0 \in \mathcal{B}(h_0)$, h_0 is the initial space, and the solution is to be studied in $h_0 \otimes \Gamma \equiv \mathcal{H}$. The coeffecients L, K, F, G are adapted, continuous (strongly) processes.

If the coefficients are constant and bounded then one can construct the solution by Cauchy-Picard type of iteration process and show that the solution is unique. If one assumes furthermore that

$K = -L^* S$, $F = S - I$, $G = \left(-\frac{1}{2} L^* L + iH\right)$ with

S unitary in $\mathfrak{h}$ and H bounded selfadjoint, then the unique solution X is also unitary in $\mathfrak{h} \otimes \Gamma$. More generally, the solutions may be bounded even if the coefficients are bounded; and for unbounded coefficients there are some results known.

Some generalisations of the above structure: for ex. one may take $\Gamma = \Gamma\left(L^2(\mathbb{R}_+, k)\right)$ with k a separable Hilbert space (of finite or infinite dimensions) — this is comparable to having a countable family of independent SBM's.

~~There is also some kind of converse result available (ref. 5); Let $\{U_{s,t}\ (s \le t)\}$ be a family of unitary~~

(*) Most of these results can be found in the books of Parthasarathy, and of Goswami - Sinha.

Let G be the gen. of a C_0-contraction semigroup $P(t)$ in h_0 and let $D(L_j) \supseteq D(G)$ $(j=1,2)$ satisfying for $u, v \in D(G)$

$$\langle v, Gu\rangle + \langle Gv, u\rangle + \sum_{j=1}^{2} \langle L_j v, L_j u\rangle = 0.$$

Spse furthermore that $\exists$ a s.a. op C and a core $\mathcal{D}$ for G s.t.

(i) $D(G) = D(C)$ & $\forall u \in D(G)$ $\exists \{u_n\} \in \mathcal{D}$ $\ni$ $u_n \to u$, $Gu_n \to Gu$ & $Cu_n \to Cu$.

(ii) $\exists$ a +ve s.a. Φ s.t. $D(\Phi) \supseteq \mathcal{D}$, $\forall u \in \mathcal{D}$, $-2\,\mathrm{Re}\langle u, Gu\rangle = \langle u, \Phi u\rangle \leq \langle u, Cu\rangle$

(iii) $L_j(\mathcal{D}) \subseteq D(C)$, $\forall j$

(iv) $\exists b > 0$ s.t. $\forall u \in \mathcal{D}$

$$2\,\mathrm{Re}\langle Cu, Gu\rangle + \sum_{j=1}^{2} \langle L_j u, C L_j u\rangle \leq b \langle u, Cu\rangle$$

i.e. formally,

$$\mathcal{L}(C) \leq bC.$$

Then U_t is unitary in $h_0 \otimes \Gamma$.
(Chebotarev - Fagnola)

In this context $G = \frac{1}{2}\sum L_j^* L_j + iH$

$$= \frac{b-1}{2} - \frac{b}{2}\left(p^2 + x^2\right) + i\left(p^2 + V(x)\right).$$

$$= \left(\frac{b}{2} - i\right)\Delta + \left(-\frac{b}{2}x^2 + iV(x) + \frac{b-1}{2}\right).$$

Is this the generator of C_0-contraction?

Cannarsa & Vespri (Siam J. Math. Anal., vol 18, 1987) helped. For V continuous and $|V(x)| \leq \text{Const.}(x^2+1)$ for $|x| > R$, $\overline{G}$ is generates a contraction (analytic) semigroup.

With $\mathcal{D} = H^{(2)}$, the L^2-Sobolev space of order 2 and $\Phi = C = b(p^2 + x^2)$, all the hypotheses of Fagnola–Chebotarev can be verified to give U_t unitary.

Next we want to look at the Heisenberg picture, i.e. the evolution of the observables x and p. For $B \in \mathcal{B}(L^2(\mathbb{R}))$, we set $j_t(B) \equiv U(t)\, B\, U(t)^*$, where U is the solution of the "Stochastic Schrödinger" or Hudson-Parthasarathy eqn above. Then it is straightforward to verify that

$$dj_t(B) = \sum_{k=1}^{2} \{ j_t([L_k^*, B])\, dA_k - j_t([L_k, B])\, dA_k^+ \} + j_t(\mathcal{L}(B))\, dt,$$

with initial value $j_0(B) = B \otimes I$ in $\mathcal{B}(L^2(\mathbb{R})) \otimes \Gamma$, and

$$\mathcal{L}(B) = \sum_{k=1}^{2} \left(L_k^* B L_k - \frac{1}{2} B L_k^* L_k - \frac{1}{2} L_k^* L_k B \right) + i[H, B],$$

the Lindblad action. From this for $B = x$ or p,

$$dj_t(x) = -i\sqrt{b}\,(dA_1 - dA_1^+) + j_t(-bx + p)\, dt,$$

$$dj_t(p) = \sqrt{b}\,(dA_1 + dA_1^+) + b^2 \{ j_t(xL_2^*)\, dA_2 + j_t(xL_2)\, dA_2^+ \} + \left(-b\, j_t(x) + b^2 j_t(x) - V'(j_t(x))\right) dt.$$

If we set the x-fluctuation and p-fluctuation as

$$F^{(x)}(t) = -i\sqrt{b}\,(A_1(t) - A_1^+(t)) \quad \text{and}$$

$$F^{(p)}(t) = \sqrt{b}\,(A_1(t) + A_1^+(t)) + b^2 \int_0^t \left(j_t(xL_2^*)\, dA_2(t) + j_2(L_2 x)\, dA_2^+(t) \right),$$

then $$\langle dF^{(x)}(t)\, dF^{(x)}(s)\rangle_{\text{Vacuum}} = b\,\delta(t-s)\,dt\,ds,$$

or equivalently for $f, g \in C_c^\infty(\mathbb{R})$ if we set

$$F^{(x)}(f) = \int_0^\infty f(t)\, dF^{(x)}(t),$$ we shall have

or

$$\langle F^{(x)}(f)\, F^{(x)}(g)\rangle_{\text{vac}} = b \int_0 f(t)\, g(t)\, dt.$$

Similarly, one gets

$$\langle dF^{(p)}(t)\, dF^{(p)}(s)\rangle_{\text{vac}} = b\,\delta(t-s)\,dt\,ds \; +$$

$$b^4 \langle j_t(x^2)\rangle_{\text{vac}}\, \delta(t-s)\,dt\,ds,$$ and

$$\langle dF^{(p)}(t)\, dF^{(x)}(s)\rangle_{\text{vac}} = i b\,\delta(t-s)\,dt\,ds,$$

$$\langle j_t(B)\rangle_{\text{vac}} = T_t(B) = e^{\mathcal{L}t}(B),$$

$\mathcal{L}$, the Lindblad map defined before.

h_{φ} = the state Hilbert sp. of the OS, $\Gamma(\partial\mathcal{H})$ is the symmetric bosonic Fock space over $\partial\mathcal{H}$, representing the environment. In our point of view these give the "Langevin" evolution replacing the canonical and Schrodinger evolutions for non-dissipative isolated systems.

Here we'll look at only quantum description of both OS and of E.

In the earlier treatment, the Fock spaces employed to model E was the so-called zero-temperature Fock spaces thereby implicitly assuming that the E was at zero absolute temp. How do we model non-zero temp. E ?

$$\mathcal{F} = \Gamma(L^2(\mathbb{R}_+, \mathbb{C}^2)) \otimes \Gamma(L^2(\mathbb{R}_+, \mathbb{C}^2))$$
$$\equiv \Gamma_1 \otimes \Gamma_2 \text{, tensor product of 2 copies.}$$

The vacuum vector $\tilde{\Omega} = \Omega_1 \otimes \Omega_2$ and the Weyl operator $\tilde{W}(f) = W_1(\alpha f) \otimes W_2(-\beta \bar{f})$, where α, β are real positive constants with $\alpha^2 - \beta^2 = 1$. The finite (non-zero) temperature property is encoded by setting:

$$\alpha = \sqrt{\frac{\exp(\varepsilon/kT)}{\exp(\varepsilon/kT) - 1}}, \quad \beta = \frac{1}{\sqrt{\exp(\varepsilon/kT) - 1}}, \quad \cdots (1)$$

k = Boltzman's const., T = absolute temp., ε a typical energy level of E (may be set = $\hbar\omega$, the lowest energy level of E).

The annihilation & creation Ops:

$$\left.\begin{aligned} \tilde{A}(f) &= \alpha A^{(1)}(f) \otimes I^{(2)} + \beta I^{(1)} \otimes A^{(2)\dagger}(f) \\ \tilde{A}^{\dagger}(f) &= \alpha A^{(1)\dagger}(f) \otimes I^{(2)} + \beta I^{(1)} \otimes A^{(2)}(f) \end{aligned}\right\} \cdots (2)$$

The commutation relation can be easily seen as,

$$[\tilde{A}(f), \tilde{A}^{\dagger}(g)] = \alpha^2 [A^{(1)}(f), A^{(1)\dagger}(g)] + \beta^2 [A^{(2)\dagger}(f), A^{(2)}(g)]$$
$$= (\alpha^2 - \beta^2) \langle f, g \rangle = \langle f, g \rangle, \text{ where } f, g \quad \cdots (3)$$

are real-valued.

If we set $\tilde{A}_j(dt) = \tilde{A}(\chi_{dt} \otimes e_j)$, then

(here e_1, e_2 is an O.N. basis for $\mathbb{C}^2$)

$$\tilde{A}_j(t) = \alpha\, A_j(t) \otimes I_2 + \beta\, I_1 \otimes A_j^{+}(t)$$

so that the Ito formula changes to

$$d\tilde{A}_j(t)\, d\tilde{A}_k^{+}(t) = \alpha^2 \delta_{jk}\, dt \quad \text{and}$$

$$d\tilde{A}_j^{+}(t)\, d\tilde{A}_k(t) = \beta^2 \delta_{jk}\, dt,$$ in contrast to the zero-temp. case.

Now we consider the following diff$\underline{\ell}$ eq$\underline{n}$ for the evolution operator:

$$d\tilde{U}(t) = \tilde{U}(t)\Big[\sum_{j=1}^{2}\Big\{L_j^{*}\, d\tilde{A}_j(t) - L_j\, d\tilde{A}_j^{+}(t) + \Big(-\frac{\alpha^2}{2} L_j^{*}L_j - \frac{\beta^2}{2} L_j L_j^{*}\Big)dt\Big\} + iH\,dt\Big],$$

$$\tilde{U}(0) = I. \qquad \cdots (8)$$

Existence and unitarity of the soln has not yet been proven, though "formally" $\tilde{U}(t)$ is unitary.

It is interesting to find out the evolution of the observables x and p.

$\tilde{j}_t(x) = \tilde{U}_t (x \otimes I) \tilde{U}_t^*$ to find:

$$d\tilde{j}_t(x) = \left[-i\sqrt{b}\,(d\tilde{A}_1 - d\tilde{A}_1^+)\right] + \left(-b\,\tilde{j}_t(x) + \tilde{j}_t(p)\right)dt$$

$$d\tilde{j}_t(p) = \left[\sqrt{b}\,(d\tilde{A}_1 + d\tilde{A}_1^+) + b^2\left(\tilde{j}_t(xL_2^*)\,d\tilde{A}_2 + \tilde{j}_t(L_2 x)\,d\tilde{A}_2^+\right)\right]$$
$$+ \left[-b\,\tilde{j}_t(p) + b^2\tilde{j}_t(x) - V'(\tilde{j}_t(x))\right]dt .$$

If we set the heat bath variables apart, by

$$d\tilde{F}^{(x)}(t) = -i\sqrt{b}\,(d\tilde{A}_1 - d\tilde{A}_1^+)$$
$$d\tilde{F}^{(p)}(t) = \left[\sqrt{b}\,(d\tilde{A}_1 + d\tilde{A}_1^+) + b^2(\quad\quad)\right],$$

then the vacuum expectation of the above fluctuations turn out to be:

$$\langle d\tilde{F}^{(x)}(t)\, d\tilde{F}^{(x)}(s)\rangle = b(\alpha^2+\beta^2)\,\delta(t-s)\,ds\,dt$$

$$\langle d\tilde{F}^{(p)}(t)\, d\tilde{F}^{(p)}(s)\rangle = \left[b + b^4\langle\tilde{j}_t(x^2)\rangle\right]\delta(t-s)\,ds\,dt$$

$$\langle d\tilde{F}^{(p)}(t)\, d\tilde{F}^{(x)}(s)\rangle = ib\,(\alpha^2 - \beta^2)\,\delta(t-s)\,ds\,dt.$$

Now $\alpha^2+\beta^2 = \coth(\mathcal{E}/2KT)$ and setting $\mathcal{E} = \hbar\omega$

$$\lim_{\hbar\to 0} \langle d\tilde{F}^{(2)}(t)\, d\tilde{F}^{(2)}(s)\rangle = \left(\frac{2KTb}{\omega}\right)\delta(t-s)\, ds\, dt$$

classical limit.

δ-correlation comes from White noise modeling of the heat bath. What is interesting is the term $\left(\frac{2KTb}{\omega}\right)$, which has temperature T and dissipation or friction coeff. b. Such results are called by Physicists & Engineers as Fluctuation-Dissipation theorems, i.e. the fluctuations in the heat bath variables contain information about the dissipative effects on the observed system.

$$\frac{dx(t)}{dt} = -\alpha x + p\,, \quad \frac{dp(t)}{dt} = -\alpha p + \alpha^2 x - V'(x)$$

or equivalently

$$\frac{d^2x(t)}{dt^2} + 2\alpha \frac{dx(t)}{dt} + V'(x) = 0,$$ the expected equation. --- (8)

Remarks:

(1) Usual treatment of heat bath in the independent oscillator model (Ford-Lewis-O'Connell), one gets in the socalled Ohmic limit the Langevin eqn:

$$\frac{d^2x}{dt^2} + 2\alpha \frac{dx}{dt} + V'(x) = \sigma \frac{dw(t)}{dt},$$ the RHS is like "white noise". We can solve this explicitly (when $V = 0$) and verify that $[x(t), p(t)] = e^{-2\alpha t}[x(0), p(0)] = i\, e^{-2\alpha t}$.

So one ~~believes~~ finds that Langevin-type model ~~should~~ does not restore the conservativity of the total system.

(2) A similar model can be worked out for a particle in the presence of gyroscopic forces (e.g. Lorentz force in E-M field case). For static electric field,

For static and uniform magnetic field and with no sources, we obtain an equation very similar to (8) with Lorentz force present. For this one has to introduce 3 quantum Brownian motions to deal with 3 physical dimensions and a 4th one to compensate for the extra terms arising out of friction.